Point vortices: From Models of Atoms to Patterns in Bose-Einstein Condensates

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Helmholtz's 1858 paper

Über Integrale der hydro-dynamischen Gleichungen, welche den Wirbelbewegungen entsprechen. Journal für reine und angewandte Mathematik 55, 25-55.

Positions: $z_\alpha = x_\alpha + iy_\alpha$; $\alpha = 1, \ldots, N$

Circulations or "strengths": Γ_α

$$\dot{z}^*_\alpha = \frac{1}{2\pi i} \sum_{\beta=1}^{N}{}' \frac{\Gamma_\beta}{z_\alpha - z_\beta}$$

Asterisk means complex conjugation; prime on summation means $\beta \neq \alpha$

Kirchhoff's 1877 lectures

Point vortex equations in Hamiltonian form:

$$\Gamma_\alpha \dot{x}_\alpha = \frac{\partial H}{\partial y_\alpha} \qquad \Gamma_\alpha \dot{y}_\alpha = -\frac{\partial H}{\partial x_\alpha}$$

where H is the Hamiltonian:

$$H = -\frac{1}{4\pi} \sum_{\alpha,\beta=1}^{N}{}' \; \Gamma_\alpha \Gamma_\beta \, log\left| z_\alpha - z_\beta \right|$$

Integrals of the motion

From translational invariance: $X + iY = \sum_{\alpha=1}^{N} \Gamma_\alpha z_\alpha$

From rotational invariance: $I = \sum_{\alpha=1}^{N} \Gamma_\alpha |z_\alpha|^2$

From invariance to time translation: H itself

Poisson bracket algebra

Definition of the Poisson bracket:

$$\left[f, g\right] = \sum_{\alpha=1}^{N} \frac{1}{\Gamma_\alpha}\left(\frac{\partial f}{\partial x_\alpha}\frac{\partial g}{\partial y_\alpha} - \frac{\partial f}{\partial y_\alpha}\frac{\partial g}{\partial x_\alpha}\right)$$

Fundamental Poisson brackets:

$$\left[z_\alpha, z_\beta\right] = 0 \qquad \left[z_\alpha, z_\beta^*\right] = -\frac{2i\delta_{\alpha\beta}}{\Gamma_\alpha}$$

Key results: $[X, Y] = \sum_{\alpha=1}^{N} \Gamma_\alpha$, $[X, I] = 2Y$, $[Y, I] = -2X$

Thus, $\left[X^2 + Y^2, I\right] = 2X\left[X, I\right] + 2Y\left[Y, I\right] = 0$

(E. Laura, 1904)

Gröbli (1877): Explicit reduction to quadratures (arbitrary circulations) of three-vortex problem

Poincaré (1893): Three integrals in involution: H, I and X^2+Y^2. Three-vortex problem is integrable for arbitrary set of vortex strengths!

Synge (1949): Geometrical interpretation of Gröbli's solutions*

*) Published in Can. J. Math. Phys. vol. 1 – same issue as Einstein & Infeld, "On the Motion of Particles in the General Theory of Relativity."

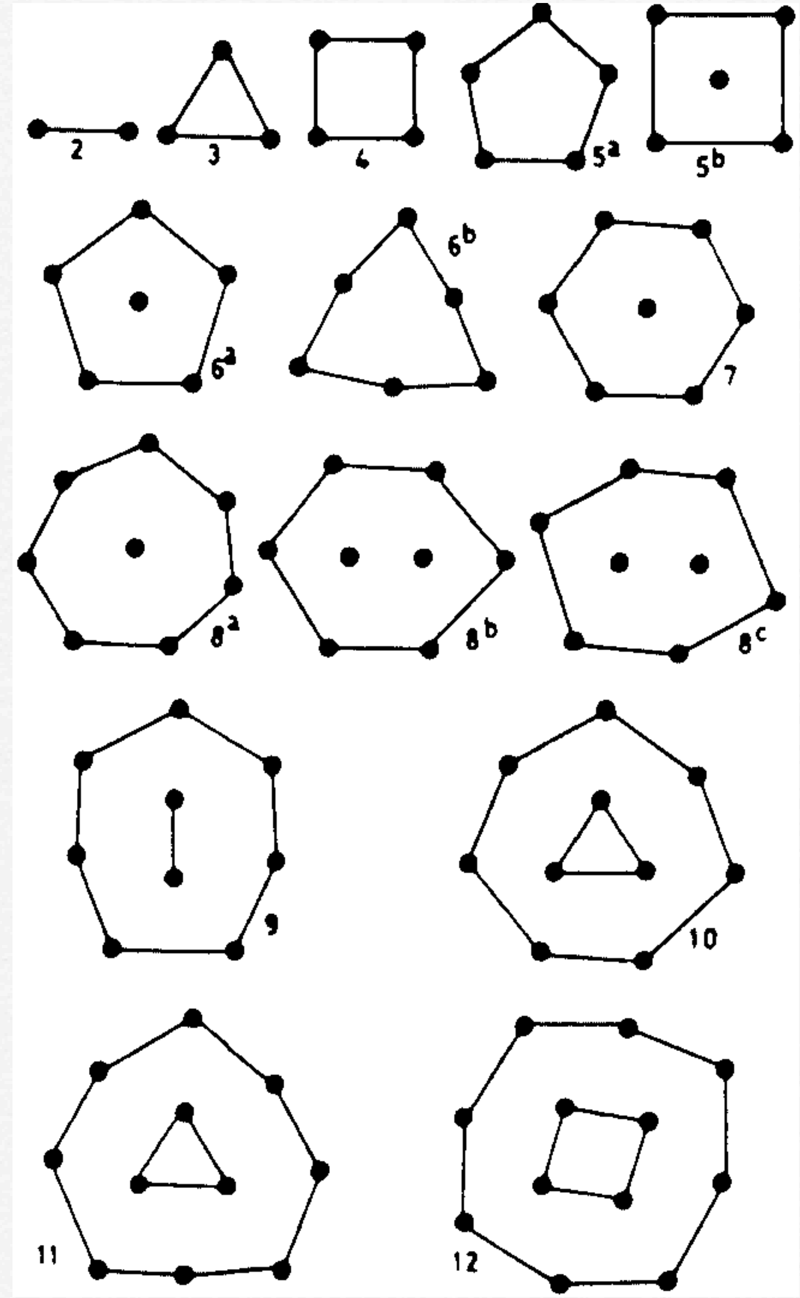
Walter Gröbli (1852-1903)

Zweites Semester. Von 24. April 1876 bis 15. August 1876

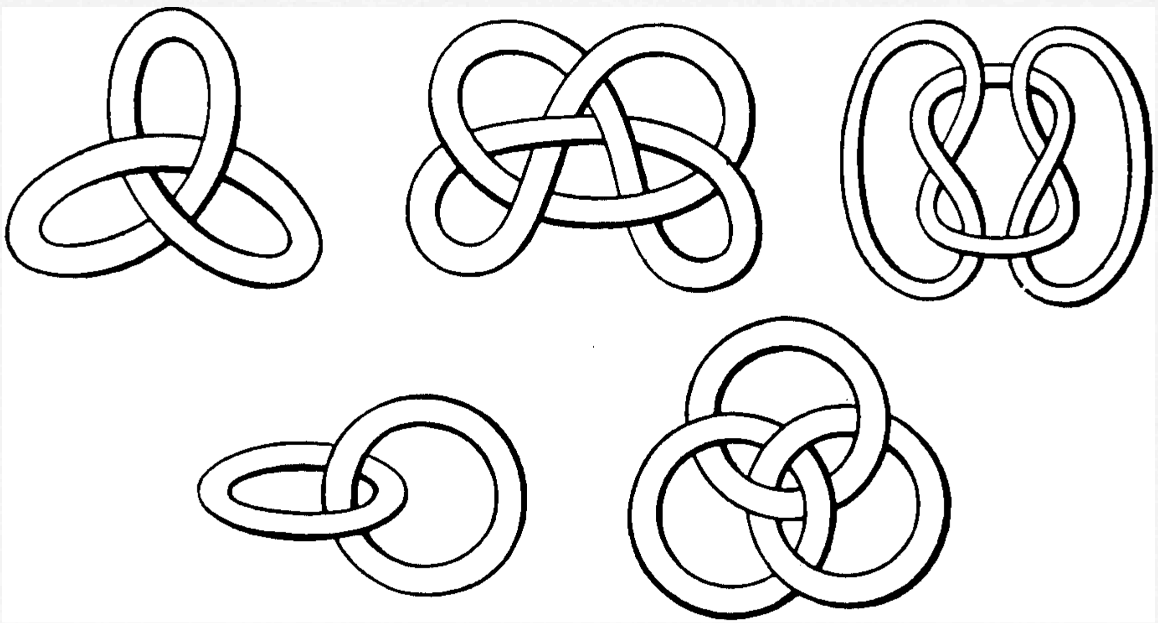
Vorlesungen.	Vermerk des Quästors betreffend das Honorar.	Nummer des Platzes im Auditorio.	Eigenhändige Einzeichnung des Docenten.	Datum der Anmeldung.	Abgemeldet bei dem Docenten.	Datum der Abmeldung.
1. Prof. G. Kirchhoff Theorie d. Wärme.	1 bezahlt	32	G. Kirchhoff	6/5 76.	G. Kirchhoff	26/7 76.
2. Prof. Helmholtz Electrodynamik	2	24	Helmholtz	4/5 76	Helmholtz	24/7 76
3. Prof. Helmholtz Die log. Principien d. Erfahrungswiss.	3 publice	38				
4. Prof. Weierstrass Ergänzungen zur Theorie d. Abel'schen F.	4 [illegible]		Weierstraß	3.5 76	Weierstraß	24/7 76
5. Prof. Kummer Principien der Wahrscheinlichkeitsr.	5 4/5 76	85	Kummer	17/5 76.	Kummer	26/7 76
6. Prof. Weierstrass Theorie d. anal. Functionen	6 bezahlt ~~[illegible]~~ 20/5 76	62	Weierstraß	22.5 76	Weierstraß	24/7 76.

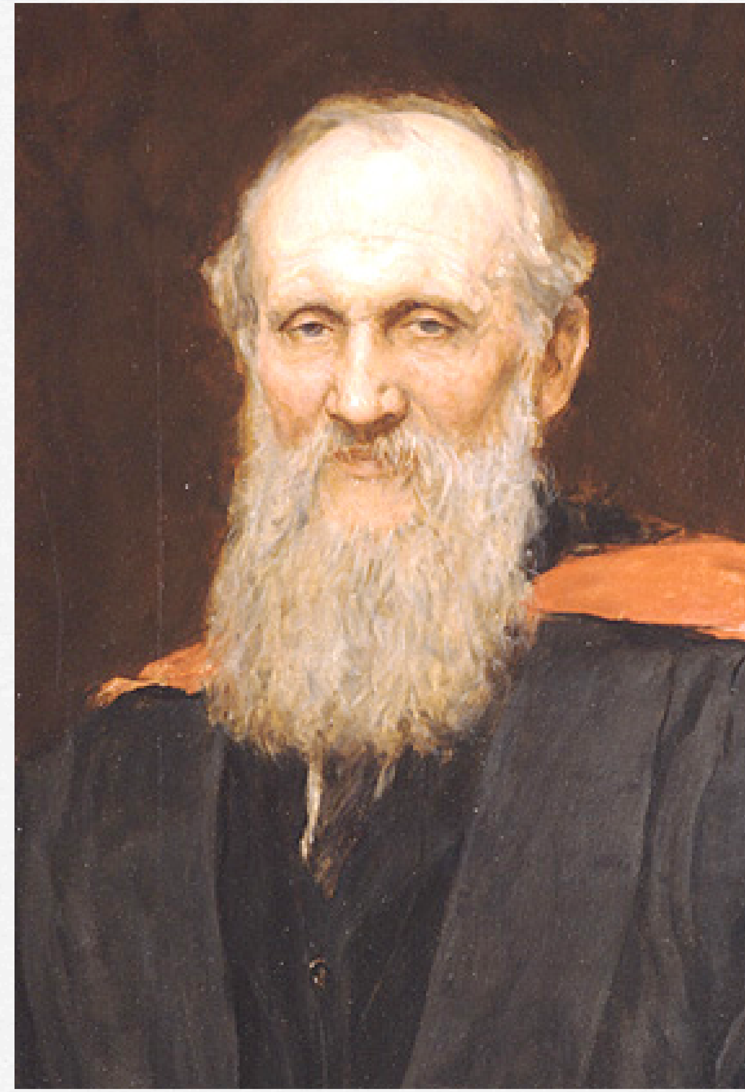
Alfred Marshall Mayer
1836-1897

A note on experiments with floating magnets... Amer. J. Sci. Arts 15, 276-277 (1878) [Also: Floating magnets. Nature 17, 487-488.]



"the mode of experimenting … described [by Mayer], with slight modification, gives a perfect mechanical illustration … of the kinetic equilibrium of groups of columnar vortices revolving in circles round their common center of gravity."





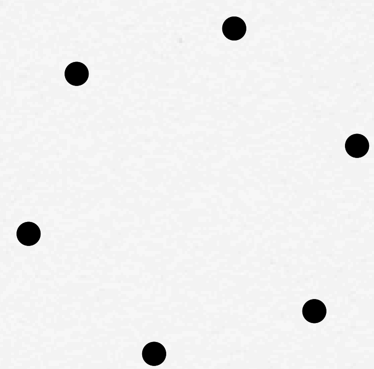
William Thomson, Lord Kelvin
1824-1907

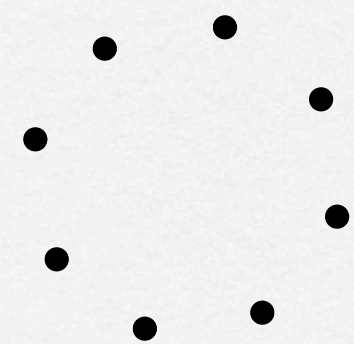
"Helmholtz's vortex rings are the only true atoms" (1869)

J. J. Thomson
1856–1940

Adams Prize Essay
"A Treatise on the Motion of Vortex Rings."
London: Macmillan (1883)

"Thomson's theorem": Regular vortex N-gon is linearly stable for $N \leq 6$, marginally stable for $N = 7$, unstable for $N > 8$.





"If we regard the system of magnets as a model of an atom, the number of magnets being proportional to the atomic weight … we should have something quite analogous to the periodic law…"
J. J. Thomson, "Cathode rays." Phil. Mag. 44 (1897) 293–316.

Kármán vortex street

- Steady state and linear stability analysis (1912)

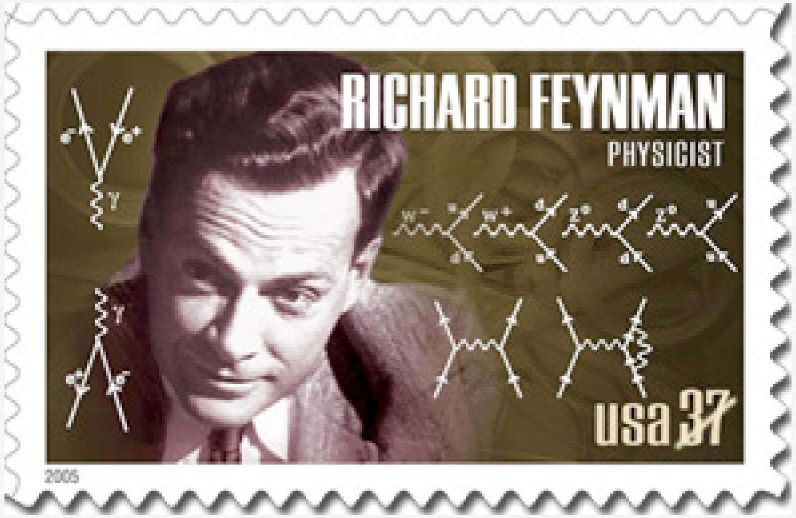
Vortex Street Near Madeira Island

26 April 2002

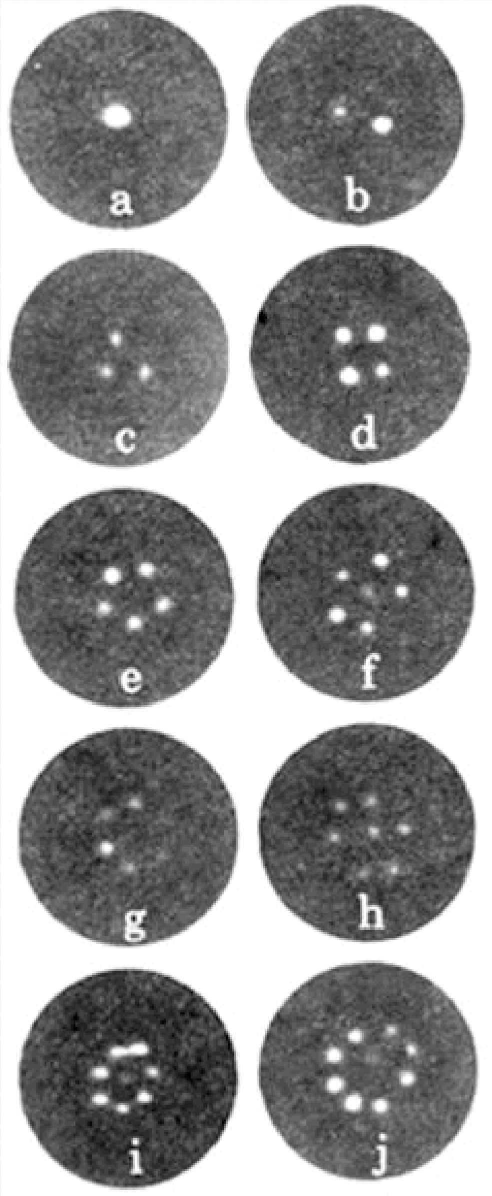
Inset: Laboratory vortex street

1940's: Onsager and Feynman predict quantized vortices in superfluid He

First visualized by
Yarmchuk, Gordon & Packard
Phys. Rev. Lett. 43 (1979)



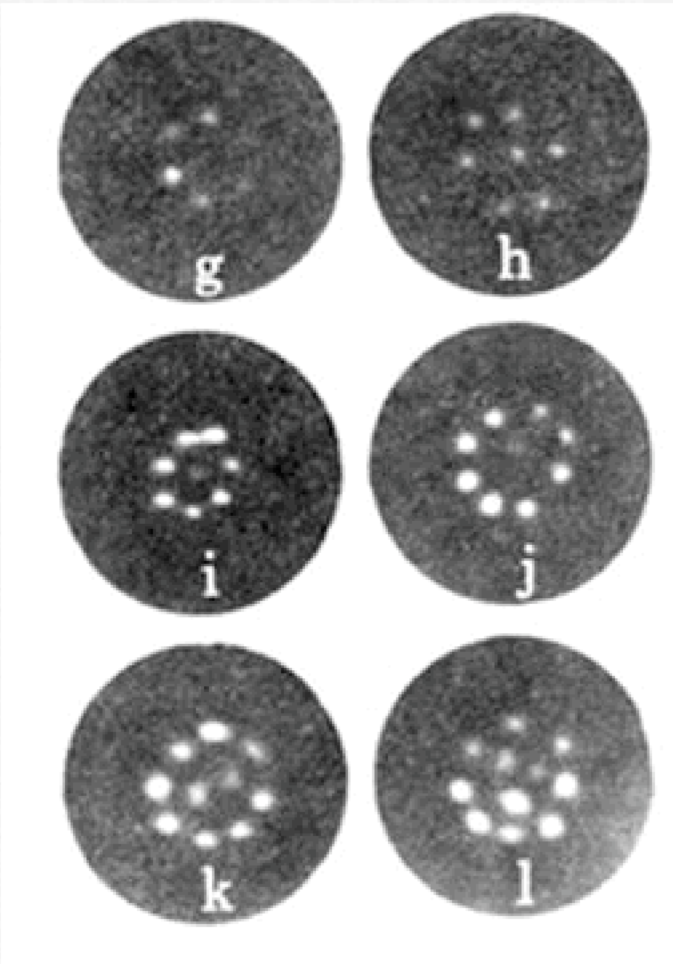
Richard Feynman
1918–1988



Lars Onsager
1903–1976

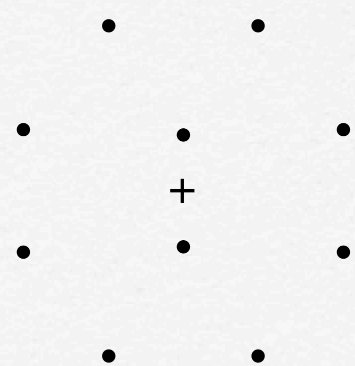
"Quantum of circulation" is h/m

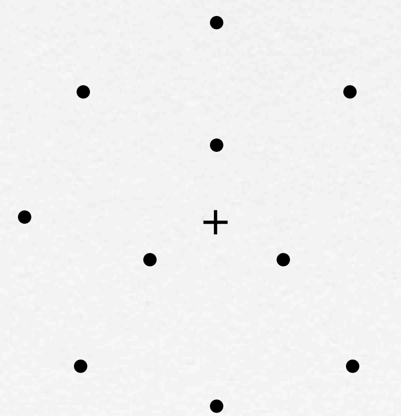
More complex relative equilibria



Solve N complex, algebraic equations:

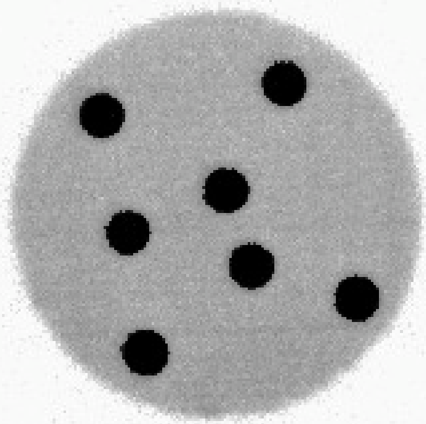
$$z^*_\alpha = {\sum_{\beta=1}^{N}}' \frac{1}{z_\alpha - z_\beta}$$



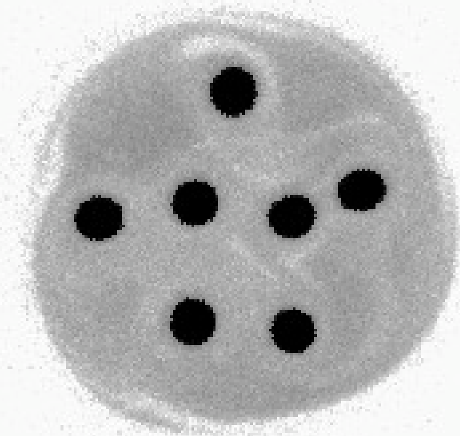


Original photos kindly provided by Dr. Durkin

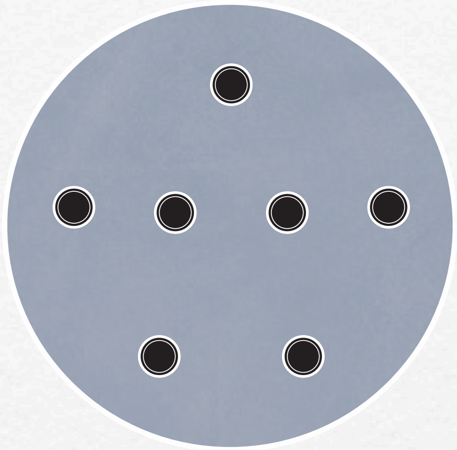
Plasma experiment



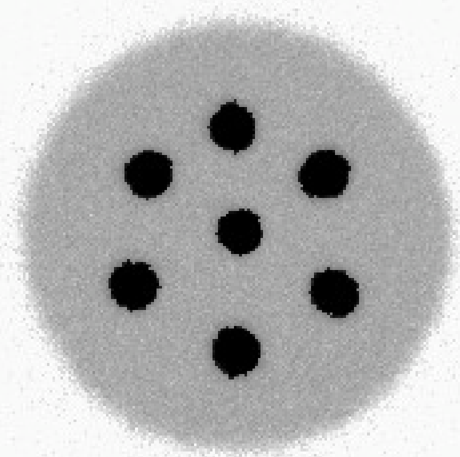
Initial condition



Metastable state



point vortex equilibrium



Asymptotic equilibrium

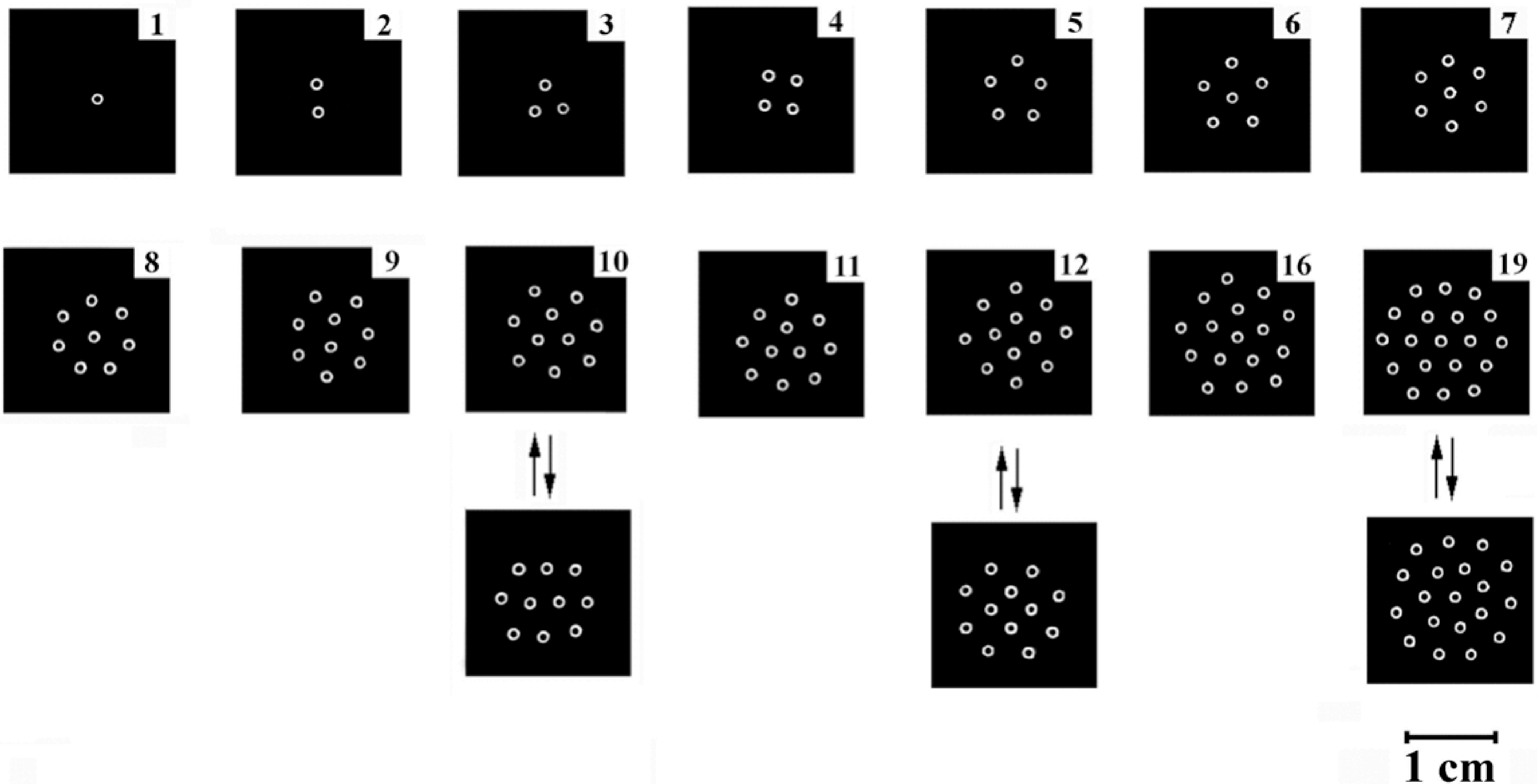
Durkin & Fajans

Physics of Fluids 12 (2000) 289

Physical Review Letters 85 (2000) 4052

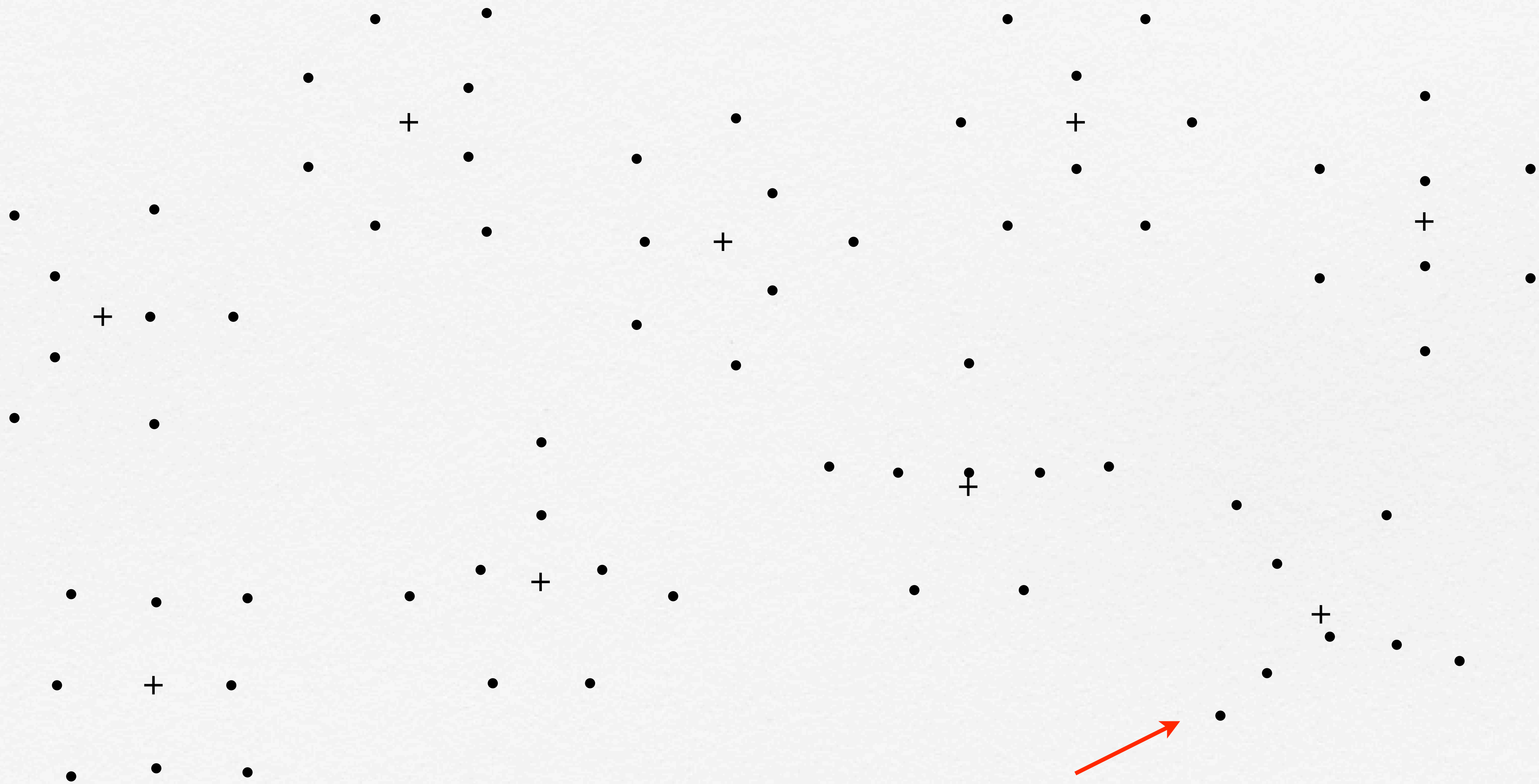
Physics Today (January 2001) p.9

“Self-assembly”



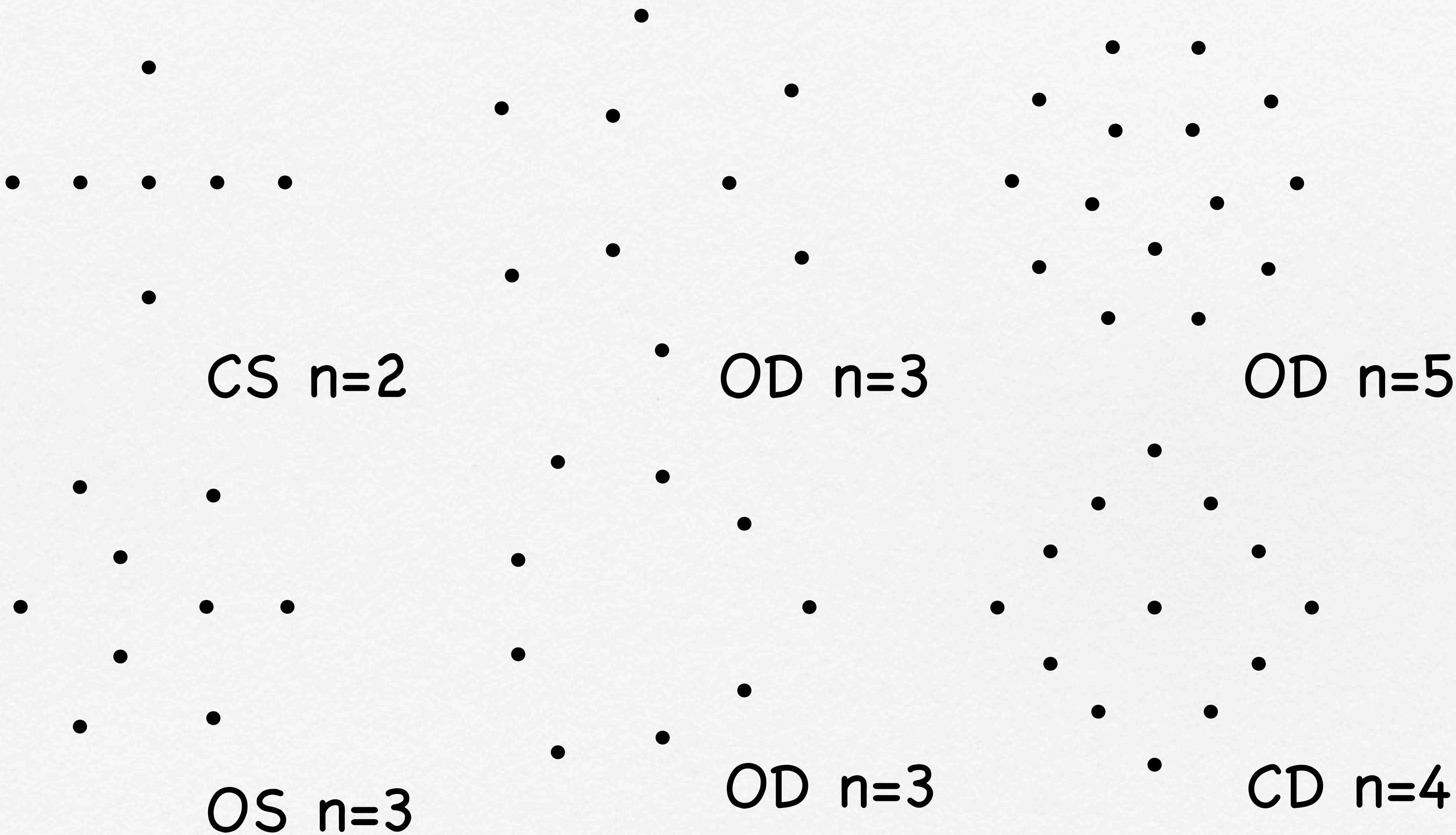
B. A. Grzybowski, H. A. Stone & G. M. Whitesides, “Dynamic self-assembly of magnetized, millimetre-sized objects rotating at a liquid-air interface.” Nature 405 (2000) 1033-1036

"Non-trivial" relative equilibria of 8 identical point vortices



Smallest known asymmetric state
Aref & Vainchtein, Nature (1998)

Analytically understood nested n-gon states

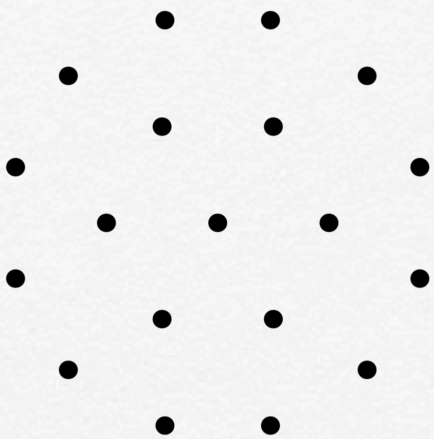


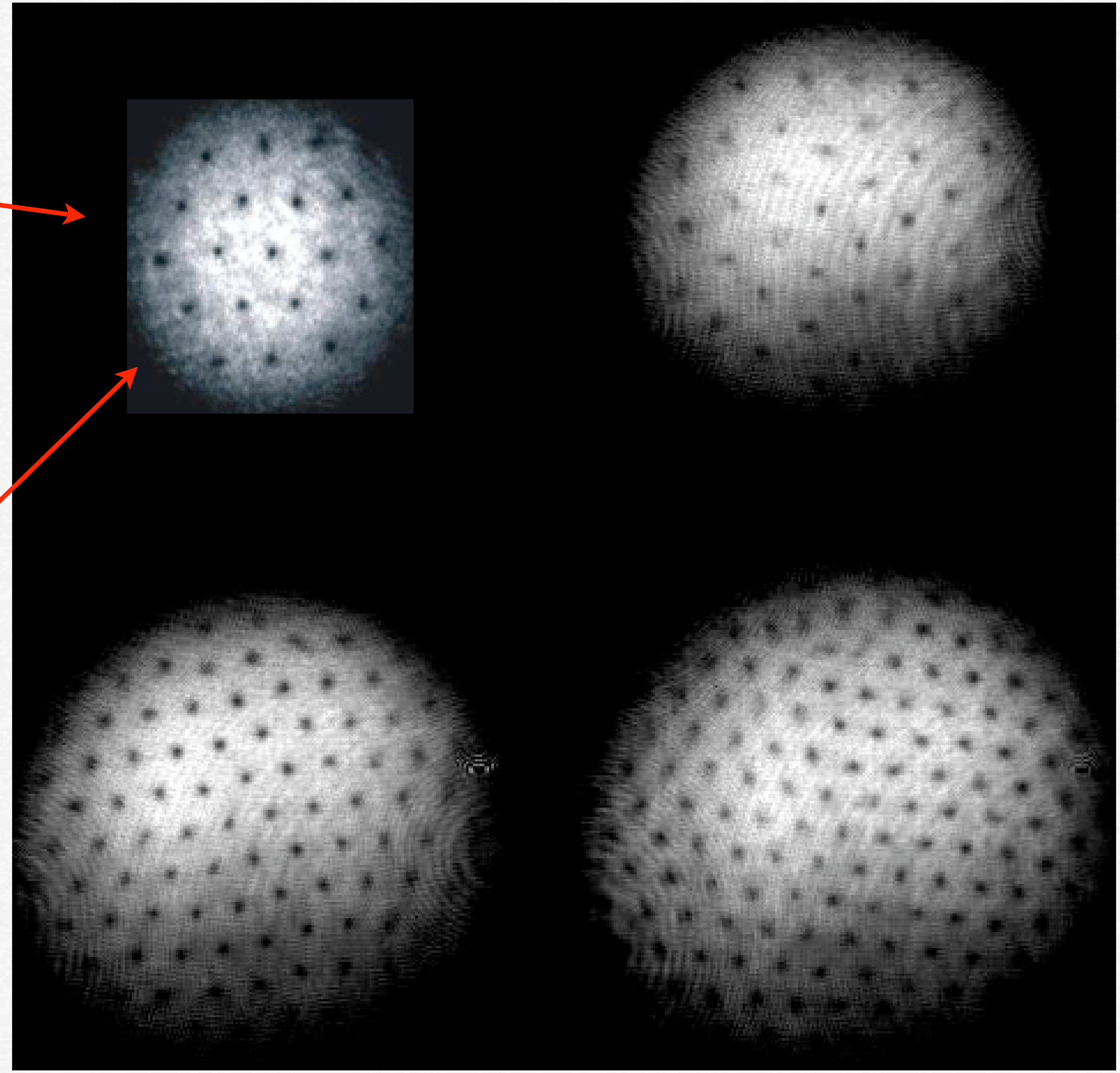
Aref & van Buren, Phys. Fluids (2005)

C-centered, O-open, S-staggered, D-degenerate

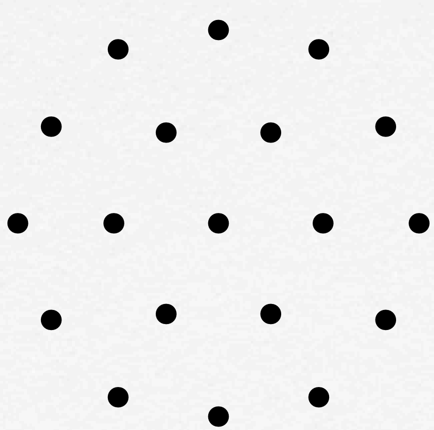
Degenerate case

BEC vortices Phys. Rev. A 70 (2004) 063607





Centered,
triple hexagons



Staggered case

Conclusions

- Point/line vortices have a varied and colorful 150-year history
- At times they have been perceived as being at the forefront of physical theory
- There are physical systems that realize - or come close to realizing - point/line vortices, e.g., electron plasma, superfluids and BECs
- Equilibria of interacting point vortices is of wider interest as a paradigm of pattern formation in related physical systems